Advanced Algorithm

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Lecture 3: Balls and Bins

Useful Inequalities

- Randomized Algorithm - Chapter 3.2 (P45)

Theorem (Union Bound)

Let E_i be a random event, then we have $Pr[\cup_{i=1}^{n} E_i] \leq \sum_{i=1}^{n} Pr(E_i)$.

Theorem (Markov Inequality)

Let Y be a random variable assuming only non-negative values. Then for all $t > 0$, $Pr[Y \geq t] \leq \frac{E[Y]}{t}$.

Theorem (Chebyshev's Inequality)

Let X be a random variable with expectation μ_X and standard deviation σ_X, then for any $t > 0$,

$$Pr[|X - \mu_X| \geq t\sigma_X] \leq \frac{1}{t^2}.$$

Useful Inequalities

- Randomized Algorithm - Chapter 4.1 (P67)

Theorem (Chernoff's Bound(1))

Let $X_1, X_2, \cdots, X_n$ *be independent Poisson trials such that, for* $1 \le i \le n$, $Pr[X_i = 1] = p_i$, *where* $0 < p_i < 1$. *Then, for* $X = \sum_{i=1}^{n} X_i, \mu = E[X] = \sum_{i=1}^{n} p_i$, *and any* $\delta > 0$,

$$Pr[X > (1+\delta)\mu] < [\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}]^{\mu}.$$

Theorem (Chernoff's Bound(2))

$\cdots$ *and any* $0 < \delta < 1$,

$$Pr[X < (1-\delta)\mu] < [\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}]^{\mu}.$$

Useful Inequalities

- Randomized Algorithm - Chapter 4.1 (P67)

Theorem (Chernoff's Bound(3))

Let $X_1, X_2, \cdots, X_n$ be independent Poisson trials such that, for $1 \le i \le n, Pr[X_i = 1] = p_i$, where $0 < p_i < 1$. Then, for $X = \sum_{i=1}^{n} X_i, \mu = E[X] = \sum_{i=1}^{n} p_i$, and any $0 < \delta < 1$,

$$Pr[|X - \mu| > \delta\mu] < 2e^{-\frac{\delta^2}{3}\mu}.$$

Balls and Bins

- m balls, n bins. You randomly throw each ball to some bin.
- X_i: number of balls in the i-th bin.
- Let $k \triangleq \max(X_1, X_2, \cdots, X_n)$.
- Question: expectation and distribution of k?

Balls and Bins

1. $m = o(\sqrt{n})$;
   - prove $Pr(k > 1) = o(1)$.
2. $m = \Theta(\sqrt{n})$; (Birthday Paradox)
   - compute $Pr(k > 1)$ again.
3. $m = n$;
   - find suitable x, such that $Pr(k \leq x) = 1 - o(1)$.
   - $k = \Theta(\frac{\ln n}{\ln \ln n})$ w.h.p;
4. $m \geq n \ln n$ (next lecture);

Homework

1. Prove Chernoff's Bound (2)(3).
2. Prove $(\frac{m}{x})^x \leq \binom{m}{x} \leq (\frac{em}{x})^x$. Here x is an integer ranged from $[1 \cdots m]$. (Stirling's formula: $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$)
3. In case 3 in balls and bins problem, let Y_i be the random event that there are more than $\frac{\ln n}{3 \ln \ln n}$ balls in the i-th bin. Prove $\forall i \neq j, Cor(Y_i, Y_j) < 0$.